Hyperfast travel in general relativity

S. V. Krasnikov*
The Central Astronomical Observatory at Pulkovo, St. Petersburg 196140, Russia
(Received 22 July 1997; published 12 March 1998)

The problem is discussed of whether a traveler can reach a remote object and return sooner than a photon would when taking into account that the traveler can partly control the geometry of his world. It is argued that under some reasonable assumptions in globally hyperbolic space-times the traveler cannot hasten reaching the destination. Nevertheless, it is perhaps possible for the traveler to make an arbitrarily long *round-trip* within an arbitrarily short (from the point of view of a terrestrial observer) time. [S0556-2821(98)03906-X]

PACS number(s): 04.20.Gz, 04.20.Cv

I. INTRODUCTION

Everybody knows that nothing can move faster than light. The regrettable consequences of this fact are also well known. Most of the interesting or promising candidates for colonization are so distant from us that the light barrier seems to make an insurmountable obstacle for any expedition. It is, for example, 200 pc from us to the Polar star, 500 pc to Deneb, and $\sim$10 kpc to the center of the Galaxy, not to mention other galaxies (hundreds of kiloparsecs). It makes no sense to send an expedition if we know that thousands of years will elapse before we receive its report. On the other hand, the prospects of being confined forever to the Solar system without any hope of visiting other civilizations or examining closely black holes, supergiants, and other marvels are so gloomy that it seems necessary to search for some way out.

In the present paper we consider this problem in the context of general relativity. Of course the light barrier exists here too. The point, however, is that in general relativity one can try to change the time necessary for some travel not only by varying one's speed but also, as we shall show, by changing the distance one is to cover.

To make the question more specific, assume that we emit a beam of test particles from the Earth to Deneb (the event S). The particles move with all possible (sub)luminal speeds and by definition do not exert any effect on the surrounding world. The beam reaches Deneb (with the arrival time of the first particle $t_{\mathcal{D}}$ by Deneb's clocks), reflects there from something, and returns to the Earth. Denote by $\Delta\tau_{\mathcal{E}}$ ($\tau_{\mathcal{E}}$ is the Earth's proper time) the time interval between S and the return of the first particle (the event R). The problem of interstellar travel lies just in the large typical $\Delta\tau_{\mathcal{E}}$. It is conceivable of course that a particle will meet a traversible wormhole leading to Deneb or an appropriate distortion of space shortening its way (see [1] and example 4 below), but one cannot hope to meet such a convenient wormhole each time one wants to travel (unless one makes them oneself, which is impossible for the *test* particles). Suppose now that instead of emitting the test particles we launch a spaceship (i.e., something that does act on the surrounding space) in S. Then the question we discuss in this paper can be formulated as follows: Is it possible that the spaceship will reach Deneb and then return to the Earth in $\Delta\tau' < \Delta\tau_{\mathcal{E}}$? By ''possible'' we mean ''possible, at least in principle, from the causal point of view.'' The use of tachyons, for example, enables, as is shown in [1], even a nontachyonic spaceship to hasten its arrival. Suppose, however, that tachyons are forbidden (as well as all other means for changing the metric with violating what we call below ''utter causality''). The main result of the paper is the demonstration of the fact that even under this condition the answer to the above question is positive. Moreover, in some cases (when global hyperbolicity is violated) even $t_{\mathcal{D}}$ can be lessened.

II. CAUSAL CHANGES

A. Changes of space-time

In this section we make the question posed in the Introduction more concrete. As the point at issue is the effects caused by *modifying the (four-dimensional) world* (that is, by changing its metric or even topology), one may immediately ask, Modifying from what? To clarify this point first note that though we treat the geometry of the world classically throughout the paper (that is, we describe the world by a space-time, i.e., by a smooth Lorentzian connected globally inextendible Hausdorff manifold) no special restrictions are imposed for a while on matter fields (and thus on the right-hand side of the Einstein equations). In particular, *it is not implied that the matter fields (or particles) obey any specific classical differential equations.*

Now consider an experiment with two possible results.

Example 1. A device set on a spaceship first polarizes an electron in the y direction and then measures the x component of its spin σ_x. If the result of the measurement is $\sigma_x = +1/2$, the device turns the spaceship to the right; otherwise it does not.

Example 2. A device set on a (very massive) spaceship tosses a coin. If it falls on the reverse the device turns the spaceship to the right; otherwise it does not.

Comment. One could argue that example 2 is inadequate since (due to the classical nature of the experiment) there is actually *one* possible result in that experiment. That is indeed the case. However, let us (i) assume that before being tossed the coin had never interacted with anything and (ii) neglect the contribution of the coin to the metric of the world. Of course, items (i) and (ii) constitute an *approximation* and

*Electronic address: redish@pulkovo.spb.su

situations are conceivable for which such an approximation is invalid [e.g., item (i) can be illegal if the same coin was already used a lot in another experiment involving large masses]. We do not consider such situations. At the same time if (i) and (ii) are adopted, experiment 2 can well be considered as an experiment with two possible results (the coin, in fact, is unobservable before the experiment).

Both situations described above suffer some lack of determinism (originating from the quantum indeterminism in the first case and from coarsening the classical description in the second). Namely, the spaceship is described now by a body whose evolution is not fixed uniquely by the initial data (in other words, its trajectory (nonanalytic, though smooth) is no longer a solution of any ''good'' differential equation). However, as stated above, this does not matter much.

So, depending on which result is realized in the experiment (all other factors being the same; see below) our world must be described by one of *two different space-times*. It is the comparison between these two space-times that we are interested in.

Notation. Let M_1 and M_2 be two space-times with a pair of inextendible timelike curves $\mathcal{E}_i, \mathcal{D}_i \subset M_i$ in each (throughout the paper $i,j=1,2$). One of these space-times, M_1 say, describes our world under the assumption that we emit test particles at some moment $S_1 \in \mathcal{E}_1$ and the other under the assumption that instead of the particles we launch a spaceship in S_2, where $S_2 \in M_2$ corresponds in a sense (see below) to S_1. The curves $\mathcal{E}_i$ and $\mathcal{D}_i$ are the world lines of Earth and Deneb, respectively. We require that the two pairs of points exist:

$$F_i \equiv \mathrm{Bd}\,[J^+(S_i)] \cap \mathcal{D}_i, \quad R_i \equiv \mathrm{Bd}\,[J^+(F_i)] \cap \mathcal{E}_i. \tag{1}$$

These points mark the restrictions posed by the light barrier in each space-time. Nothing moving with a subluminal speed in the world M_i can reach Deneb sooner than in F_i or return to Earth sooner than in R_i. What we shall study is just the relative positions of S_i, F_i, R_i for $i=1,2$ when the difference in the space-times M_1 and M_2 is of such a nature (below we formulate the necessary geometrical criterion) that it can be completely ascribed to the pilot's activity after S.

B. ''Utter causality''

The effect produced by the traveler on space-time need not be weak. For example, by a (relatively) small expenditure of energy the spaceship can break the equilibrium in some close binary system on its way, thus provoking the collapse. The causal structures of M_1 and M_2 in such a case will differ radically. If an advanced civilization (to which it is usual to refer) will cope with topology changes, it may turn out that M_1 and M_2 are even nondiffeomorphic. So the space-times being discussed may differ considerably. On the other hand, we want them to be not *too* different.

(a) The pilot of the spaceship deciding in S whether or not to fly to Deneb knows the pilot's past and in our model we would prefer that the pilot's decision could not change this past. This restriction is not incompatible with the fact that the pilot can make different decisions (see the preceding subsection).

(b) The absence of tachyons (i.e., fields violating the postulate of local causality [2]) does not mean in itself that one (located in, say, point A) cannot act on events lying off one's ''causal future'' [i.e., off $J^+(A)$]:

(i) Matter fields are conceivable that while satisfying local causality themselves do not provide local causality to the metric. In other words, they afford a unique solution to the Cauchy problem for the metric, not in $D^+(\mathcal{P})$ (cf. Chap. 7 in [2]), but in some smaller region only. In the presence of such fields the metric at a point B might depend on the fields at points outside $J^-(B)$. That is, the metric itself would act as a tachyon field in such a case.

(ii) Let M_1 be the Minkowski space with coordinates (t_1, x_1^μ) and M_2 be a space-time with coordinates (t_2, x_2^μ) and with the metric flat at the region $x_2^1 > t_2$, but nonflat otherwise (such a space-time describes, for example, propagation of a plane electromagnetic wave). Intuition suggests that difference between M_1 and M_2 cannot be ascribed to the activity of an observer located in the origin of the coordinates, but neither local causality nor any other principle of general relativity forbids such an interpretation.

In the model we construct we want to abandon any possibility of such ''acausal'' action on the metric. In other words, we want the condition relating M_1 and M_2 to imply that these worlds are the same in events that cannot be causally connected to S. This requirement can be called the *principle of utter causality*.

C. Relating condition

In this section we formulate the condition relating M_1 to M_2. Namely, we require that M_i should ''diverge by S'' (see below). It should be stressed that from the logical point of view this condition is just a *physical postulate.* Being concerned only with the relation between two ''possible'' worlds, this new postulate does not affect any previously known results. In defense of restrictions imposed by our postulate on M_i we can say that it does not contradict any known facts. Moreover, in the absence of tachyons [in the broad sense, see item (i) above] it is hard to conceive of a mechanism violating it.

Formulating the condition being discussed, we would like to base it on the ''principle of utter causality.'' In doing so, however, we meet a circle: To find out whether a point is causally connected to S_i we must know the metric of the space-time M_i, while the metric at a point depends in turn on whether or not the point can be causally connected to S_i. That is why we cannot simply require that $M_i \setminus J^+(S_i)$ be isometric. The following example shows that this may not be the case even when utter causality apparently holds.

Example 3: ''Hyperjump.'' Let M_1 be the Minkowski plane with S_1 located at the origin of the coordinates and let M_2 be the space-time (similar to the Deutsch-Politzer space) obtained from M_1 by the following procedure (see Fig. 1). Two cuts are made, one along a segment l lying in $I^+(S_1)$ and another along a segment l' lying off $J^+(S_1)$ and obtained from l by a translation. The four points bounding l, l' are removed and the lower (or the left, if l is vertical) bank of each cut is glued to the upper (or to the right) bank of the other. Note that we can vary the metric in the shadowed region without violating utter causality though this region ''corresponds'' to a part of $M_1 \setminus J^+(S_1)$.

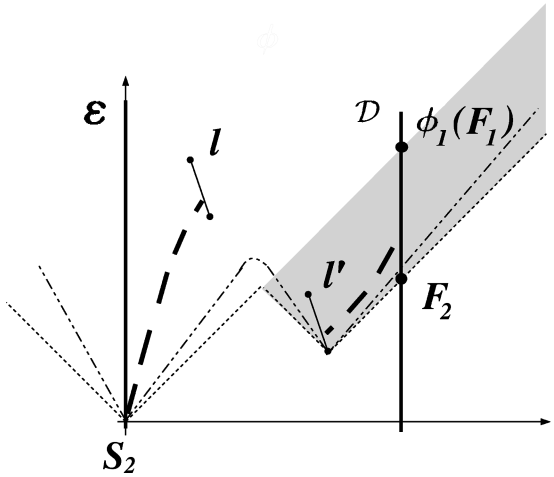

FIG. 1. ''Hyperjump.'' The thick dashed line depicts an allowed world line of a spaceship.

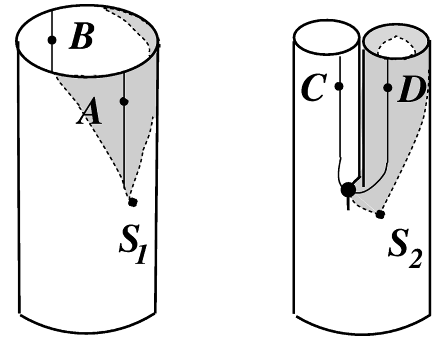

FIG. 2. Make cuts along the thin lines on the cylinder at the left and glue their banks to obtain the ''trousers'' at the right. The shadowed regions depict $J^+(S)$. Note that these space-times cannot be considered as diverging by S. If we take, for example, the whole M_i with the thin lines removed, as N_i, then Bd $N_1 \ni B \star C$, while neither B nor C lies in $\mathbb{J}$.

To overcome this circle we shall formulate our relating condition in terms of the boundaries of the ''unchanged'' regions.

Notation. Below we deal with two regions $N_i \subset M_i$ related by an isometry ϕ: $N_2 = \phi(N_1)$. To shorten notation we shall write sometimes $X_{(1)}$ for a subset $X \subset N_1$ and $X_{(2)}$ for $\phi(X)$. The notation $A \star B$ for points A,B will mean that there exists a sequence $\{a_n\}$,

$$a_{n(i)} \to A, \quad a_{n(j)} \to B.$$

Clearly if $A \in N_1$, then $A \star B \neq A$ means simply $B = \phi(A)$. Finally, $\mathbb{J} \equiv \overline{J^+(S_1)} \cup \overline{J^+(S_2)}$.

Definition 1. We call space-times M_1, M_2 *diverging by the event* S_1 (or *by* S_2 or simply *by* S) if there exist open sets $N_i \subset M_i$, points S_i, and an isometry ϕ: $N_1 \mapsto N_2$ such that $I^-(S_2) = \phi[I^-(S_1)]$ and

$$(Q_j \cup Q_k) \cap \mathbb{J} \neq \varnothing \tag{2}$$

whenever $Q_j \in$ Bd N_j and $Q_j \star Q_k \neq Q_j$.

Comment. In the example considered above the two space-times diverged by S. Note the following. (i) The possible choice of N_i is not unique. The dotted lines in Fig. 1 bound from above two different regions that can be chosen as N_2. (ii) $A_{(2)} \prec B_{(2)}$ does not necessarily imply $A_{(1)} \prec B_{(1)}$. (iii) Points constituting the boundary of N fall into two types, some have counterparts (i.e., points related to them by $\star$) in the other space-time and the others do not (corresponding thus to singularities). It can be shown (see Lemma 1 in the Appendix) that the first type points form a dense subset of Bd N.

In what follows we proceed from the assumption that the condition relating the two worlds is just that they are described by space-times diverging by S (with N_i corresponding to the unchanged regions). It should be noted, however, that this condition is tentative to some extent. It is not impossible that some other conditions may be of interest, more restrictive than ours (e.g., we could put some requirements on points of the second type) or, on the contrary, less restrictive. The latter can be obtained, for example, in the following manner. The relation $\star$ is reflective and symmetric, but not transitive. Denote by $\sim$ its transitive closure (e.g., in the case depicted in Fig. 2, $A \not\star B$, but $A \sim B$). Now, if we want to consider topology changes like that in Fig. 2 as possibly produced by the event S, we can replace Eq. (2) by the requirement that for any first type point $Q \in$ Bd N_j,

$$[Q]_{\sim} \cap \mathbb{J} \neq \varnothing, \tag{3}$$

where $[Q]_{\sim} \equiv \{x | x \sim Q\}$. It is worth pointing out that replacing Eq. (2) by Eq. (3) does not actually affect any of the statements below.

Now we can formulate the question posed in the Introduction as follows: Given that the space-times M_i diverged by an event S, how will the points F_2, R_2 be related to the points F_1, R_1? [It is understood from now on that $\mathcal{C}_2 \cap N_2 = \phi(\mathcal{C}_1 \cap N_1)$, where $\mathcal{C}_i = \mathcal{D}_i$, $\mathcal{E}_i$.]

III. ONE-WAY TRIP

Example 3 shows that contrary to what one might expect, utter causality by itself does not prevent a pilot from hastening the arrival at a destination. It is reasonable to suppose, however, that in less ''pathological'' space-times[1] this is not the case.

Proposition 1. If M_i are globally hyperbolic space-times diverging by S, then

$$F_1 \star F_2.$$

The proof of this seemingly self-evident proposition has turned out to be quite tedious, so we cite it in the Appendix.

Example 4. Recently it was proposed [1] to use for hyperfast travel the metric (I omit two irrelevant dimensions y and z)

$$ds^2 = -dt^2 + [dx - v_s f(r_s) dt]^2. \tag{4}$$

[1]Note that we discuss the causal structure only. So the fact that there are singularities in the space-time from example 3 is irrelevant. As is shown in [3], a singularity-free space-time can be constructed with the same causal structure.

Here $r_s \equiv |x - x_s|$, $v_s(t) \equiv dx_s(t)/dt$, and $x_s(t)$ and f are arbitrary smooth functions satisfying[2]

$$x_s(t) = \begin{cases} D & \text{at } t > T \\ 0 & \text{at } t < 0, \end{cases}$$

$$f(\xi) = \begin{cases} 1 & \text{for } \xi \in (-R+\delta, R-\delta) \\ 0 & \text{for } \xi \notin (-R, R) \ . \end{cases}$$

δ, T, and R are arbitrary positive parameters.

To see the physical meaning of the condition of utter causality take the Minkowski plane as M_1 and the plane endowed with the metric (4) as M_2 (we choose the origins to be S_i). It is easy to see that the curve $\lambda \equiv [t, x_s(t)]$ is timelike with respect to the metric (4) for any $x_s(t)$. So we could conclude that an astronaut can travel with an arbitrary velocity [''velocity'' here is taken to mean the coordinate velocity $dx_a(t)/dt$, where $x_a(t)$ is the astronaut's world line]. All one needs is to choose an appropriate $x_a(t)$ and to make the metric be of form (4) with $x_s(t) = x_a(t)$. The distortion of the space-time in the region $\{0 < x < D, \ t > 0\}$ of M_2 will allow the astronaut to travel faster than one could have done in the flat space M_1 (which does not of course contradict Proposition 1 since the M_i do not diverge by S).

The subtlety lies in the words ''to make the metric be . . . .'' Consider the curve $\lambda_+ \equiv [t, x_s(t) + R]$, which separates the flat and the curved regions. It is easy to see that $v_s(t) > 1$ when and only when $\lambda_+(t)$ is spacelike. At the same time Eq. (19) of [1] says that the space immediately to the left of λ_+ is filled with some matter ($G^{00} \neq 0$).[3] The curve $\lambda_+(t)$ is thus the world line of the leading edge of this matter. We come therefore to the conclusion that to achieve $T < D$ the astronaut has to use tachyons. This possibility is not too interesting: no wonder that one can overcome the light barrier if one can use the tachyonic matter. Alternatively, in the more general case, when the space-time is nonflat from the outset, a similar result could be achieved without tachyons by placing *in advance* some devices along the pilot's way and programming them to come into operation at preassigned moments and to operate in a preassigned manner. Take the moment P when we began placing the devices as a point diverging the space-times. Proposition 1 shows then that, though a regular spaceship service perhaps can be set up by this means, it does not help to outdistance the test particles from M_1 in the *first* flight (i.e., in the flight that would start at P).

IV. ROUND-TRIP

The situation with the points R_i differs radically from that with F_i since the segment FR belongs to $J^+(S)$ for sure. So even in globally hyperbolic space-times there is nothing to prevent an astronaut from modifying the metric so as to move R closer to S (note that from the viewpoint of possible applications to interstellar expeditions this is far more important than to shift F). Let us consider two examples.

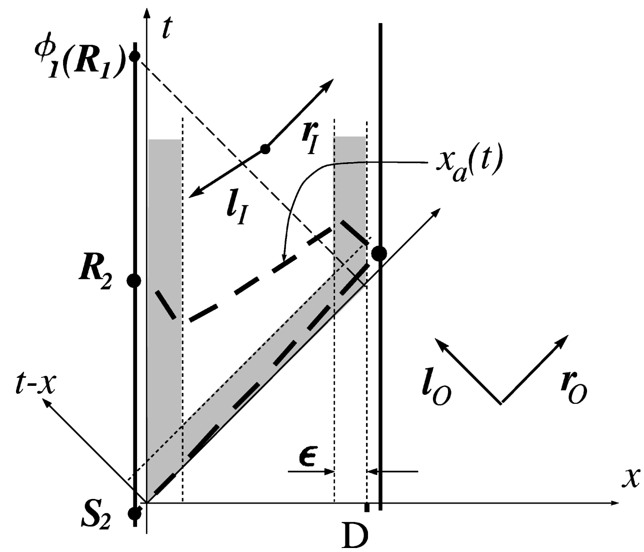

FIG. 3. Warp drive.

Example 5: ''The warp drive.'' Consider the metric

$$ds^2 = -(dt - dx)[dt + k(t,x)dx],$$

where $k \equiv 1 - (2-\delta)\theta_\epsilon(t-x)[\theta_\epsilon(x) - \theta_\epsilon(x + \epsilon - D)]$. Here θ_ϵ denotes a smooth monotone function

$$\theta_\epsilon(\xi) = \begin{cases} 1 & \text{at } \xi > \epsilon \\ 0 & \text{at } \xi < 0, \end{cases}$$

δ and $\epsilon < D$ being arbitrary small positive parameters.

Three regions can be recognized in M (see Fig. 3): *the outside region* $\{x<0\} \cup \{x>D\} \cup \{x>t\}$, in which the metric is flat ($k=1$) and future light cones are generated by vectors $\mathbf{r}_O = \partial_t + \partial_x$ and $\mathbf{l}_O = \partial_t - \partial_x$; *the transition region*, which is a narrow (of width $\sim\epsilon$) strip shown as a shaded region in Fig. 3 in which the space-time is curved; and *the inside region* $\{x < t - \epsilon\} \cap \{\epsilon < x < D - \epsilon\}$, which is also flat ($k = \delta - 1$), but the light cones are ''more open'' here being generated by $\mathbf{r}_I = \partial_t + \partial_x$ and $\mathbf{l}_I = -(1-\delta)\partial_t - \partial_x$. The vector $\mathbf{l}_I$ is almost antiparallel to $\mathbf{r}_I$ and thus a photon moving from F toward the left will reach the line $x=0$ almost in S.

We see thus that an arbitrarily distant journey can be made in an arbitrarily short time. It can look like the following. In 2000, say, an astronaut, whose world line is shown as a bold dashed line in Fig. 3, starts to Deneb. The astronaut moves with a near light speed and the way to Deneb takes the (proper) time $\Delta\tau_a \ll 1600$ yr. On the way he or she carries out some manipulations with the ballast or with the passing matter. In spite of these manipulations the traveler reaches Deneb at 3600 only. However, on the way back the traveler finds that the metric has changed and he or she moves ''backward in time,'' that is, t decreases as Earth is approached (though the traveler's trajectory, of course, is *future directed*). As a result, the traveler returns to Earth in 2002.

Example 6: Wormhole. Yet another way to return arbitrarily soon after the start by changing geometry is the use of wormholes. Assume that we have a wormhole with a negligibly short throat and with both mouths resting *near Earth.* Assume further that we can move any mouth at will without

[2] In [1] another f was actually used. Our modification, however, in no way impairs the proposed spaceship.

[3] The case in point is, of course, a four-dimensional space.

changing the ‘‘inner’’ geometry of the wormhole. Let the astronaut take one of the mouths with him or her. If he or she moves with a near light speed, the trip will take only the short time $\Delta\tau_a$ for the traveler. According to our assumptions, the clocks on Earth as seen through the throat will remain synchronized with the astronaut’s and the throat will remain negligibly short. So, if immediately after reaching Deneb he or she returns to Earth through the wormhole’s throat, it will turn out that he or she will have returned within $\Delta\tau_{\mathcal{E}}\approx\Delta\tau_a$ after the start.

Similar things were discussed many times in connection with the wormhole-based time machine. The main technical difference between a time machine and a vehicle under consideration is that in the latter case the mouth only moves *away* from Earth. So causality is preserved and no difficulties arise connected with its violation.

V. DISCUSSION

In all examples considered above the pilot, roughly speaking, ‘‘transforms’’ an ‘‘initially’’ spacelike (or even past-directed) curve into a future-directed curve. Assume now that one applies this procedure first to a spacelike curve (AC_1B) and then to another spacelike curve (BC_2A) lying in the intact, until then, region. As a result one obtains a closed timelike curve (AC_1BC_2A) (see [4–6] for more details). So the vehicles in discussion can be in a sense considered as ‘‘square roots’’ of time machine (and thus a collective name *space machine*—also borrowed from science fiction—seems most appropriate for them). The connection between time and space machines allows us to classify the latter under two types.

(i) The first are those leading to time machines with compactly generated Cauchy horizons (examples 4–6). From the results of [7] it is clear that the creation of a space machine of this type requires violation of the weak energy condition. The possibility of such violations is restricted by the so-called quantum inequalities, (QIs) [8]. In particular, with the use of a QI it was shown in [5] that to create a four-dimensional analog of our example 5 one needs huge amounts (e.g., $10^{32}M_{galaxy}$) of ‘‘negative energy.’’ Thermodynamical considerations suggest that this in its turn necessitates huge amounts of ‘‘usual’’ energy, which makes the creation unlikely. This conclusion is quite sensitive to the details of the geometry of the space machine and one could try to modify its construction so as to obtain more appropriate values. Another way, however, seems more promising. The QI used in [5] was derived with the constraint (see [8]) that in a region with the radius smaller than the proper radius of curvature space-time is ‘‘approximately Minkowski’’ in the sense that the energy density (to be more precise, the integral $E[\lambda,\tau_0,T]\equiv\int_{-\infty}^{T}\langle T_{\mu\nu}u^\mu u^\nu\rangle(\tau^2+\tau_0{}^2)^{-1}\,d\tau$, where λ is a timelike geodesic parametrized by the proper time τ, $\mathbf{u}\equiv\partial_\tau$, and τ_0 is a ‘‘sampling time’’) is given by essentially the same expression as in the Minkowski space. So, in designing space machines, space-times are worth searching for where this constraint breaks down.

Among them is a ‘‘critical’’ (i.e., just before its transformation into a time machine) wormhole. Particles propagating through such a wormhole again and again experience (regardless of specific properties of the wormhole [9]) an increasing blueshift. The terms in the stress-energy tensor associated with nontrivial topology also experience this blueshift [10]. As a result, in the vicinity of the Cauchy horizon (even when a region we consider is flat and is located far from either mouth) the behavior of the energy density has nothing to do with what one could expect from the ‘‘almost Minkowski’’ approximation [11]. (The difference is so great that *beyond* the horizon we cannot use the known quantum field theory, including its methods of evaluating the energy density, at all [12].) Consider, for example, the Misner space with the massless scalar field in the conformal vacuum state. From the results of Sec. III B [11] it is easy to see that $E[\lambda,\tau_0,\infty]=-\infty$ for *any* λ and τ_0 and the QI thus does not hold here.[4] Moreover, $E[\lambda,\tau_0,T]\to-\infty$ as one approaches the Cauchy horizon along λ. So we need not actually create a time machine to violate the QI. It would suffice to ‘‘almost create’’ it. Thus it well may be that in spite of (or owing to) the use of a wormhole the space machine considered in example 6 will turn out to be more realistic than that in example 5.

(ii) Then there are noncompact space machines, as in example 3. These (even their singularity free versions; see [3,13]) do not necessitate violations of the weak energy condition. They have, however, another drawback typical for time machines. The evolution of nonglobally hyperbolic space-times is not understood clearly enough and so we do not know how to *force* a space-time to evolve in the appropriate way. There is an example, however (the wormhole-based time machine [14]), where the space-time is denuded of its global hyperbolicity by quite conceivable manipulations, which gives us some hope that this drawback is actually not fatal.

ACKNOWLEDGMENTS

I am grateful to D. Coule, A. A. Grib, G. N. Parfionov, and R. R. Zapatrin for useful discussion. This work was partially supported by RFFI grant 96-02-19528.

APPENDIX

Throughout this section we take M_i to be globally hyperbolic space-times diverging by S, and $(U)_{\mathbb{J}}$ to mean $U\setminus\mathbb{J}$ for any set U.

Lemma 1. Let O be a neighborhood of a point of Bd N_j and $O^N\subset O\cap N_j$ be such an open nonempty set that

$$\mathrm{Bd}\ O^N\cap O\subset\mathrm{Bd}\ N_j. \tag{A1}$$

Then

$$\mathrm{Bd}\ O^N{}_{(i)}\cap J^+(S_i)\neq\varnothing \quad \text{for some } i.$$

Proof. Let $j=1$ for definiteness. Consider a smooth manifold $\widetilde{M}\equiv M_2\cup_{\phi'}O$, where ϕ' is the restriction of ϕ on O^N. Induce the metric on $\widetilde{M}$ by the natural projections

[4] It is most likely (see Sec. IV of [7]) that the same is true in the four-dimensional case as well.

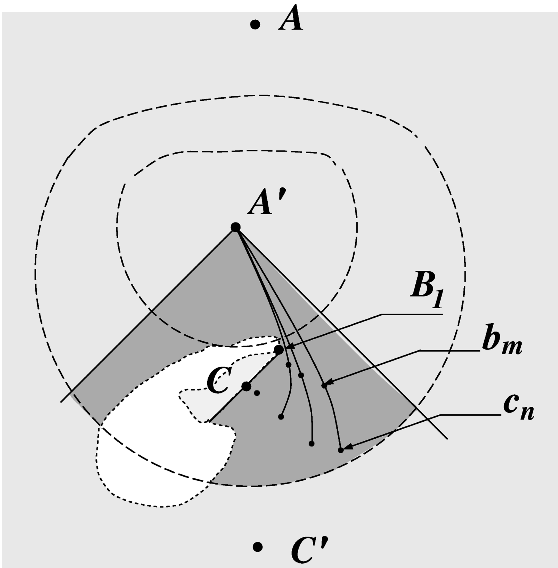

FIG. 4. Case (i) of Lemma 2. The white area does not belong to N_1 and the darkest area is W. If instead of the larger area bounded by a dashed line we take the smaller one as $V_{A'}$, we get case (ii).

$$\pi_i:\quad M_2 \overset{\pi_1}{\mapsto} \widetilde{M}, \quad O \overset{\pi_2}{\mapsto} \widetilde{M}$$

(or, more precisely, by π_i^{-1}) thus making $\widetilde{M}$ into a Lorentzian manifold and π_i into isometrical embeddings. $\widetilde{M}$ must be non-Hausdorff since otherwise it would be a space-time and so (as $M_2 \subsetneq \widetilde{M}$) M_2 would have an extension in contradiction to its definition. So points Q_i exist:

$$Q_1 \star Q_2, \quad Q_1 \in \mathrm{Bd}\, O^N_{(1)} \cap O, \quad Q_2 \in \mathrm{Bd}\, O^N_{(2)} \qquad \text{(A2)}$$

and the lemma follows now from Definition 1 coupled with Eq. (A1). ■

Lemma 2. If both $A_{(i)}$ lie in $(N_i)_{\mathbb{J}}$, then so do $I^-(A_{(i)})$.

Proof. M_i are globally hyperbolic. So any point P has such a neighborhood (we shall denote it by V_P) that, first, is causally convex, i.e., $J^-(x) \cap J^+(y) \subset V_P$ for any points x,y: $y \in J^-(x, V_P)$, and, second, lies in a convex normal neighborhood of P. Now suppose the lemma were false. We could find then such a point $A' \in I^-(A_{(i)}, N_i)$ (let $i=1$, for definiteness) that

$$W \neq I^-(A', V_{A'}),$$

where $W \equiv I^-(A', N_1 \cap V_{A'})$. Denote Bd $W \cap I^-(A', V_{A'})$ by ∂W. Clearly $\varnothing \neq \partial W \subset \overline{N_1}$. So let us consider the two possible cases (see Fig. 4).

(i) $\partial W \not\subset \mathrm{Bd}\, N_1$. Under this condition a point C and a sequence of causal curves $\{\gamma_n\}$ from A' to points c_n exist such that

$$\gamma_n \subset W, \quad c_n \to C \in \partial W \cap N_1.$$

According to [15], Proposition 2.19, there exists a causal curve γ connecting A' and C, which is limit for $\{\gamma_n\}$ and is lying thus in $\overline{W}$. Since $V_{A'}$ belongs to a normal convex neighborhood and $C \in I^-(A', V_{A'})$, γ by [2], Proposition 4.5.1 is not a null geodesic and hence

$$\gamma \not\subset N_1 \qquad \text{(A3)}$$

(otherwise by [2], Proposition 4.5.10 and by causal convexity of $V_{A'}$ we could deform it into a timelike curve lying in $N_1 \cap V_{A'}$, while $C \notin W$).

Now note that for any $C' \in I^-(C, N_1)$ there exists a subsequence $\{\gamma_k\}$ lying in $I^+(C', N_1)$. So by Eq. (A3) a sequence of points $\{b_m\}$ and a point B_1 can be found such that

$$b_m \to B_1 \in \mathrm{Bd}\, N_1, \quad b_m \in I^-(A', N_1) \cap I^+(C', N_1). \qquad \text{(A4)}$$

Thus the $\phi(b_m)$ lie in a compact set $J^-(A'_{(2)}) \cap J^+(C'_{(2)})$ and therefore

$$\phi(b_m) \to B_2: \quad B_1 \star B_2.$$

From Definition 1 it follows that at least one of the B_i lies in $J^+(S_i)$ and since $B_i \in I^-(A_{(i)})$ we arrive at a contradiction.

(ii) $\partial W \subset \mathrm{Bd}\, N_1$. In this case taking $O = I^-(A', V_{A'})$ and $O^N = W$ in Lemma 1 yields

$$\overline{W_{(i)}} \cap J^+(S_i) \neq \varnothing \quad \text{for some } i,$$

which gives a contradiction again since $\overline{W_{(i)}} \subset I^-(A_{(i)})$. ■

Consider now the sets $L_i \equiv \{x | I^-(x) \subset N_i\}$. They have a few obvious features

$$L_i = \overline{\mathrm{Int}\, L_i}, \quad \mathrm{Int}\, L_i \subset N_i, \qquad \text{(A5)}$$

$$A_{(1)} \in (L_1)_{\mathbb{J}} \quad \Leftrightarrow \quad A_{(2)} \in (L_2)_{\mathbb{J}}. \qquad \text{(A6)}$$

Combining Lemma 2 with Eqs. (A5) and (A6) we obtain

$$(\mathrm{Bd}\, L_i)_{\mathbb{J}} \subset \mathrm{Bd}\, N_i. \qquad \text{(A7)}$$

Lemma 3. $(L_i)_{\mathbb{J}} = (M_i)_{\mathbb{J}}$.

Proof. Since $(M_i)_{\mathbb{J}}$ is connected and $(\mathrm{Int}\, L_i)_{\mathbb{J}}$ is nonempty [e.g., from Definition 1 $I^-(S_i) \subset (\mathrm{Int}\, L_i)_{\mathbb{J}}$] it clearly suffices to prove that $(\mathrm{Bd}\, L_i)_{\mathbb{J}} = \varnothing$. To obtain a contradiction, suppose that there exists a point $A \in (\mathrm{Bd}\, L_1)_{\mathbb{J}}$ and let U be such a neighborhood of A that

$$\overline{U} \subset (M_1)_{\mathbb{J}}.$$

Then for $U^L \equiv U \cap \mathrm{Int}\, L_1$ it holds that

$$\overline{U^L_{(i)}} \cap J^+(S_i) = \varnothing, \quad i = 1,2.$$

On the other hand, because of Eqs. (A5) and (A7) we can take $O = U$ and $O^N = U^L$ in Lemma 1 and get

$$\overline{U^L_{(i)}} \cap J^+(S_i) \neq \varnothing \quad \text{for some } i,$$

which is a contradiction. ■

Corollary 1. $[I^+(\mathcal{E}_2)_{\mathbb{J}}] = \phi\{[I^+(\mathcal{E}_1)]_{\mathbb{J}}\}$.

Proof of Proposition 1. M_i is causally simple. Hence a segment of null geodesic from S_i to F_i exists. By [2], Proposition 4.5.10 this implies that any point $P_{(i)} \in (\mathcal{E}_i)_{\mathbb{J}}$ can be connected to F_i by a timelike curve. Hence a point $P' \in (\mathcal{D}_i)_{\mathbb{J}}$ can be reached from $P_{(i)}$ by a timelike curve without intersecting $J^+(S_i)$. Thus F_i is the future end point of the curve $\mathcal{D}'_i$:

$$\mathcal{D}'_i \equiv \mathcal{D} \cap [I^+(\mathcal{E}_i)]_{\mathbb{J}}.$$

At the same time from Corollary 1 it follows that $\phi(\mathcal{D}'_1) = \mathcal{D}'_2$. ■

[1] M. Alcubierre, Class. Quantum Grav. **11**, L73 (1994).
[2] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).
[3] S. V. Krasnikov, Class. Quantum Grav. (to be published), gr-qc/9702031.
[4] A. Everett, Phys. Rev. D **53**, 7365 (1996).
[5] A. E. Everett and T. A. Roman, Phys. Rev. D **56**, 2100 (1997).
[6] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[7] S. W. Hawking, Phys. Rev. D **46**, 603 (1992).
[8] L. H. Ford and T. A. Roman, Phys. Rev. D **53**, 5496 (1996).
[9] S. V. Krasnikov, Class. Quantum Grav. **11**, 1 (1994).
[10] U. Yurtsever, Class. Quantum Grav. **8**, 1127 (1991).
[11] S. V. Krasnikov, Phys. Rev. D **54**, 7322 (1996).
[12] B. S. Kay, M. J. Radzikowski, and R. M. Wald, Commun. Math. Phys. **183**, 533 (1997); S. V. Krasnikov, gr-qc/9802008.
[13] S. V. Krasnikov, Gen. Relativ. Gravit. **27**, 529 (1995).
[14] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
[15] J. Beem and P. Ehrich, *Global Lorentzian Geometry* (Dekker, New York, 1981).